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On the Existence of Periodic Solutions on 2-Manifolds

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1. INTRODUCTION

In this paper we shall study flows generated by a C^1 vector field on a compact two-dimensional manifold. For the sake of simplicity we shall assume that the manifold is orientable in which case the manifold is Σ^h, a sphere with h handles. We shall assume that the vector field has a finite, *positive* number of nondegenerate singularities. Here, "nondegenerate" is taken to mean that the eigenvalues associated with the linearized vector field at a singularity have nonzero real parts. The problem that then interests us is to determine when there exists a nontrivial periodic motion on Σ^h. (A singularity is defined to be a trivial periodic motion.)

DEFINITION. A *detractor* is a singularity which becomes an attractor upon reversing the vector field.

DEFINITION. Let P and Q be singularities. A *transit orbit* is an orbit $\pi(t)$ with the property that $\lim_{t\to-\infty} \pi(t) = P$ and $\lim_{t\to+\infty} \pi(t) = Q$. We say that P (or Q) *generates* the transit orbit.

Consider now the following hypotheses:

H1: There are no attracting singularities.

H2: There exists no closed curve consisting entirely of saddles and transit orbits.

The latter assumption is slightly more general than the one encountered in structural stability questions, namely, that there exist no transit orbits joining saddles. The assumption of nondegeneracy alone implies that a singularity must be either an attractor, a detractor or a saddle [4; Chap. 15].

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Thus if we assume H1 in addition, a singularity must be a detractor or a saddle.

Let us illustrate the type of result one may expect by considering a special case.

THEOREM 1.1. *Let V be a C^1 vector field on the sphere Σ^0 and assume there are a finite number of nondegenerate critical points. If* H1 *and* H2 *hold then there exists a nontrivial periodic orbit.*

Proof. The index of a nondegenerate saddle is -1 and a detractor $+1$. Since the sum of the indexes of the singularities must equal the Euler–Poincaré index of Σ^0, which is 2, there must be at least two detractors present. After removing a small neighborhood of all the detractors we obtain a compact region in the plane into which the flow is entering through the boundary. The Poincaré–Bendixon theorem then applies to give us the stated result.

Simple examples can be constructed to show that on Σ^0 neither H1 nor H2 can be deleted.

One might ask whether Theorem 1.1 is valid on Σ^h for $h \geqslant 1$. Unfortunately this is not always true. It turns out that the manner in which detractors are joined to saddles by transit orbits plays an important role in determining whether there exists a nontrivial periodic motion.

On Σ^h for $h \geqslant 1$ it is more appropriate to ask whether there exists a nontrivial recurrent motion, which, according to the Birkhoff Recurrance Theorem [1], is equivalent to seeking the existence of a compact minimal set that does not contain a singularity. We note then that if the vector field is of class C^2, then, by a result of A. J. Schwartz [2], every such nontrivial recurrent motion is either a periodic motion or a minimal invariant torus τ in which case the manifold itself must be τ. This latter case is excluded, however, by the previous assumption that singularities are present.

In the C^1 case if a nondegenerate singularity is a saddle then it is known that the stable and unstable manifolds issuing from the saddle are C^1 curves and that each manifold consists of exactly two branches [5].

Actually the nondegeneracy assumption of Theorem 1.1, and in other theorems to follow, can be relaxed, but at the expense of creating technical complications which would only obscure the ideas we wish to convey. We will concentrate on presenting some new techniques for working with vector fields and not worry about obtaining the utmost generality. For this reason we shall assume that the vector field defining the flow is of class C^1. The modifications needed for continuous vector fields and even for general continuous flows are rather straightforward. In certain applications in Section 3 we are able to perform surgery on the manifold to produce simpler flows on manifolds of lower genus, where the desired results follow immediately.

Throughout the paper we will make use of the well-known fact: *If a compact invariant set contains no singularities then it contains a nontrivial recurrent motion.*

Let $\pi(x, t)$ be the solution satisfying the initial condition $\pi(x, 0) = x$. The following two lemmas are immediate consequences of the continuity properties of a flow:

LEMMA 1.1. *Let π be a flow, K a compact set and $\{x_n\}$ a sequence $x_n \in K$ with $x_n \to x$. Consider a sequence of real numbers $t_n \to \infty$. If $\pi(x_n, t) \in K$ for $0 \leqslant t \leqslant t_n$, then $\pi(x, t) \in K$ for all $t \geqslant 0$. If, in addition, K contains no singularities, then (since the ω-limit set Ω_x lies in K) there exists a recurrent motion in K.*

LEMMA 1.2. *Let $\{y_n\}$ be a sequence $y_n \to y$ and $t_n \geqslant 0$ a sequence of reals. Suppose $z_n = \pi(y_n, t_n)$ converges, $z_n \to z$. If the t_n are bounded then there exists a subsequence, call it t_n, such that* $\lim t_n = t'$ *and $\pi(y_n, t) \to \pi(y, t)$, uniformly for $t \in [0, t']$, moreover $\pi(y, t') = z$.*

2. FLOWS WITH A FREE DETRACTOR

DEFINITION. We say that a detractor is a *free detractor* if it generates no transit orbit.

THEOREM 2.1. *Let π be a flow generated by a C^1 vector field on Σ^h with at least one free detractor, and assume* H1 *and* H2 *hold. Then there exists a nontrivial recurrent motion.*

Proof. Let P be a free detractor. Define the region of detraction corresponding to P:

$$\mathscr{D} = \{x \in \Sigma^h : \lim_{t \to -\infty} \pi(x, t) = P\}.$$

The boundary $\partial\mathscr{D}$ is compact and invariant. If it contains no singularities, then it contains a nontrivial recurrent motion.

Thus we assume there is a singularity Q in $\partial\mathscr{D}$. Clearly Q must be a saddle. Consider a neighborhood of Q as shown in Fig. 1. It follows that at least one branch of the stable manifold and at least one branch of the unstable manifold of Q must lie in $\partial\mathscr{D}$. Assume they are locally the positive x and y axes. Without any loss of generality (see next paragraph) we can assume that P lies in the first quadrant. Furthermore, we can assume that there are no

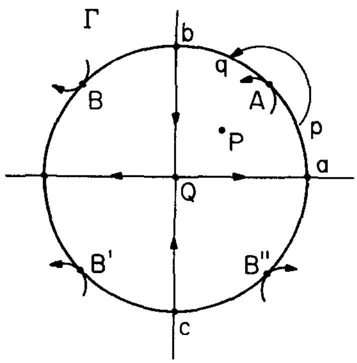

FIGURE 1

other singularities in the given neighborhood. It is also easy to see that the neighborhood can be chosen so that its boundary curve Γ has the following properties: The flow is transverse to Γ except at the point A (where the orbit through A bounces inward) and at the points B, B' and B'', where the orbits bounce outward.

In order to prove the existence of such a neighborhood we first choose small canonical neighborhoods ΔP and ΔQ of P and Q, respectively, (Fig. 1a).

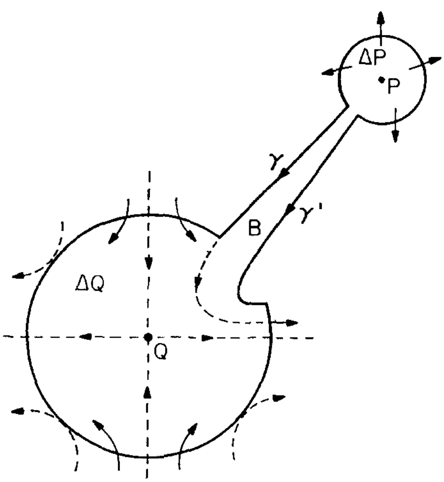

FIGURE 1a

Since Q must lie on the boundary of the region of detraction for P there is a trajectory γ that leaves P and enters ΔQ . By continuity there is another trajectory γ' near γ that also enters ΔQ . In fact there is an entire band B

of trajectories leaving P that enter ΔQ. It then follows that the region Δ bounded by the solid lines in Fig. 1a is a neighborhood of Q and by introducing appropriate coordinates P appears in the first quadrant. It only remains to adjust the boundary of Δ so that the flow has the transversality and bouncing properties illustrated in Fig. 1. Note that the shape of the neighborhood is not the essential feature but rather the distribution of bouncing points on the boundary of the neighborhood. We refer to Fig. 1 for the sake of simplicity.

Assume an ordering on Γ which increases in the counterclockwise direction with the origin at c and consider the flow leaving Γ at a point $p < A$. For p sufficiently close to A this orbit must reenter Γ at $q > A$ and, furthermore, the orbit curve l from p to q is *homotopically trivial* relative to the closed disc Δ bounded by Γ, i.e., l plus the interval $[p, q]$ on Γ is homotopic to a point. Define a mapping $T : p \to q$ in this way. T is then a homeomorphism of a sufficiently small open interval $J = (\alpha, A)$ onto $T(J) = (A, \beta)$. We now *define* T at α by setting $T\alpha = \beta$ and consider the possibility of extending T beyond α. On the one hand, if the orbit through α leaves Γ transversally and reenters transversally at β (in finite time, of course) then it is clear that T can be extended to an open interval N about α and $\beta \in T(N)$. On the other hand, either (i) the orbit through α never reaches β, or (ii) it reaches β and bounces off Γ at α (from the outside of course), or (iii) it reaches β and bounces off Γ at β. In case (ii) it is evident that $\alpha = B'' < a$ and in case (iii), that $\beta = B > b$ (see Fig. 1).

LEMMA 2.1. *If $\alpha < a$ or $\beta > b$ then there exists a periodic orbit L which bounds a disc containing P but not Q.*

Remark. This contradicts the assumption $Q \in \partial\mathscr{D}$.

Proof of lemma. If $\beta > b$ then $b \in T(J)$. Follow the orbit λ from b backward to $p = T^{-1}(b)$. By H2, $p \neq a$. (Fig. 2). If $p > a$ consider the

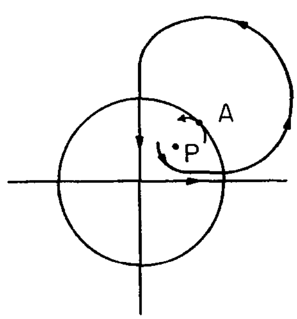

FIGURE 2

α-limit set λ^- of λ. The Poincaré–Bendixon theorem asserts that $\lambda^- = P$ or λ^- is a periodic orbit. The first possibility is ruled out since P is a free detractor. That the periodic orbit surrounds P and not Q is evident from index considerations. If $p < a$ then the ω-limit set of the orbit through a is a periodic orbit by the same reasoning. The case $\alpha < a$ is similar. This proves the lemma.

We are thus left with the case (i) in which the orbit γ through α never reaches β and $a \leqslant \alpha < A < \beta \leqslant b$. If the ω-limit set of γ, γ^+ does not contain a singularity then it certainly contains a recurrent motion. Otherwise let γ^+ contain a saddle and let $\{x_n\}$ be a sequence of points of J with $x_n \to \alpha$ monotonically and let τ_n be the time it takes for the trajectory through x_n to arrive at $y_n = Tx_n$, i.e., $y_n = \pi(x_n, \tau_n)$. The sequence $\{\tau_n\}$ must satisfy $\tau_n \to \infty$. Let γ_n represent the arc of trajectory from x_n to $y_n = Tx_n$. We now prove.

LEMMA 2.2. *The arcs γ_n converge to a curve γ joining α to β and consisting of an arc of a stable manifold joining α to a saddle, an arc of an unstable manifold joining β to a saddle, and transit orbits joining saddles.* (See Fig. 3).

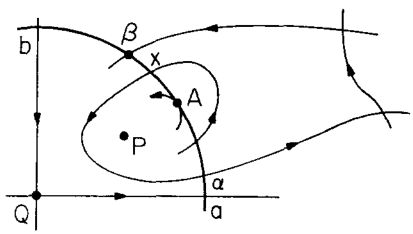

FIGURE 3

Proof of lemma. Let $Q, Q_1, \ldots, Q_N$ be all the saddle points. Choose ϵ so that the ϵ-neighborhoods $N_\epsilon(Q_i)$, $i = 1,\ldots, N$, are pairwise disjoint and in addition do not intersect Γ. Let ϵ be sufficiently small that the flow in $N_\epsilon(Q_i)$ is canonical, i.e., the orbits resemble hyperbolic arcs which bounce off the boundary of N_ϵ at exactly four points. Each neighborhood consists of four quadrants. Let $K_1, \ldots, K_{4(N+1)}$ denote the closure of these quadrants as well as the quadrants at Q bounded by Γ. Let K_1 be the quadrant containing the points x_n on its boundary.

Claim (A). For fixed n, γ_n enters each K_i at most once. This will follow from the fact that each γ_n plus the arc $\overline{y_n x_n}$ on $\Gamma \cap K_i$ bounds a disc Δ_n. If λ and λ' are two arcs of $\gamma_n \cap K_i$ for some fixed i, then let l be a line segment in K_i from λ to λ'. If l meets no other component of $\gamma_n \cap K_i$, then l lies wholly in Δ_n or in the complement $\Delta_n{}^c$. But then the flow on λ must be in the opposite direction to that on λ' which is clearly impossible. Therefore

l must meet another component of $\gamma_n \cap K_i$. Proceeding in this manner we get infinitely many components of $\gamma_n \cap K_i$ for a fixed n, which contradicts the fact that the time τ_n is finite. This proves Claim (A).

Claim (B). Let i be fixed and suppose $\gamma_{n'} \cap K_i \neq \varnothing$ for some n'. Then $\gamma_n \cap K_i \neq \varnothing$ for all $n \geqslant n'$. To prove this, let l be a line segment in K_i from a point of $\gamma_{n'}$ to the saddle s forming one of the vertices of K_i. Since the discs Δ_n form an increasing sequence, i.e., $\Delta_{n+1} \supset \Delta_n$ and s lies in none of them, l intersects all of them.

Returning now to the proof of the lemma, let ν_n be the number of K_i, $i \geqslant 2$, entered by γ_n. From (A) we see that $\nu_n \leqslant 4(N+1)$ and from (B) $\nu_{n+1} \geqslant \nu_n$. Thus there exists $\bar{n}$ such that for $n \geqslant \bar{n}$, $\nu_n = \nu =$ constant. For $n \geqslant \bar{n}$ let $K_1',\ldots, K_\nu'$ be the quadrants intersected by the γ_n listed in the order of intersection. Let p_n^j be the point at which γ_n enters K_j'. Since the Δ_n are nested, the p_n^j (j fixed) move monotonically along $\partial K_j'$ and hence converge. (Note that the p_n^j cannot lie on the portion of $\partial K_j'$ consisting of the stable and unstable manifolds of the saddle at the vertex.) If none of these sequences converge to a point on a stable manifold of a saddle, then it follows that for all n, the γ_n stay a fixed distance δ away from all saddle points contradicting the fact that γ^+, the ω-limit set of $\pi(\alpha, t)$, contains a saddle. Thus let j_0 be the smallest j for which p_n^j converges to a point q on a stable manifold, $p_n^{j_0} \to q$. Let σ_n satisfy $\pi(x_n, \sigma_n) = p_n^{j_0}$. Then $\{\sigma_n\}$ is bounded and without any loss of generality we can assume that $\{\sigma_n\}$ converges, say that $\sigma = \lim \sigma_n$. It follows then from Lemma 1.2 that $\pi(x_n, \sigma_n) = p_n^{j_0} \to q = \pi(\alpha, \sigma)$. This proves that α lies on a stable manifold of a saddle. Now by repeating the above argument we are led to a proof of Lemma 2.2.

Returning to the proof of the theorem, we see that $\alpha = a$ and $\beta = b$ would violate H2. We now follow the argument of Lemma 2.1. If $\alpha > a$, we follow the negative semi-orbit λ through α. If λ remains inside Γ it must stay away from Q since orbits near Q hit Γ near b as t decreases. The Poincaré–Bendixon theorem then implies that there exists a periodic orbit surrounding P, again contradicting the fact that $Q \in \partial\mathscr{D}$. If λ leaves Γ at $x < \beta$ (Fig. 3), then continuing λ we would arrive at $T^{-1}x \in \Gamma$ via an orbit segment which is homotopically trivial relative to Γ. Continuing at $t \to -\infty$ we see that λ is trapped in a disc with one singularity P and the Poincaré–Bendixon gives a periodic orbit surrounding P, which leads to the same contradiction. If λ leaves Γ at a point $x > \beta$ we repeat the above argument, this time using the positive semiorbit through β.

Finally if $\alpha = a$ then $\beta < b$ and the positive semiorbit through β generates a periodic orbit surrounding P. Thus in all cases we arrive at a contradiction, proving the theorem.

Remark. We have not made essential use of the fact that P is a singularity. It is clear that P can be replaced by a simple closed curve γ bounding a disc with handles across which the vector field is everywhere pointing into the annular region bounded by γ, the x axis, the y axis and the directed segment of Γ from a to b.

3. Applications

We start this section by defining a special configuration of singularities. Throughout this section we shall assume that both hypotheses H1 and H2 hold.

Definition. A *loop* is a simple closed curve consisting of saddles, detractors and transit orbits joining them so that along the curve the saddles and detractors appear in alternating order.

Of course a loop must contain the same number of detractors as saddles, but two loops may intersect in a common detractor. We prove the following basic fact concerning a loop.

Lemma 3.1. *Given a loop l and $\epsilon > 0$ there exists a σ-neighborhood of l, $G_\sigma(l)$, $\sigma \leqslant \epsilon$, which is detracting, i.e., the vectors on the boundary $\partial G_\sigma(l)$ are pointing outward.*

Proof. Let the saddles and detractors in l be denoted by s_i and d_i, $i = 1,\ldots, N$ and let $\delta < \epsilon$ be so small that $N_\delta(l) = \bigcup_{x\in l} N_\delta(x)$ is topologically an annulus. Let $\sigma \leqslant \delta$ be so small that on $\partial N_\sigma(d_i)$ the vectors are pointing outward and on $N_\sigma(s_i)$ the flow is canonical, i.e., the trajectories not on the stable and unstable manifolds resemble hyperbolic arcs. If σ is sufficiently small then the orbits starting at $t = 0$ in $N_\sigma(s_i)$ remain in $N_\delta(l)$ for all time $t \leqslant 0$. Let

$$E = \bigcup_{i=1}^{N} [N_\sigma(s_i) \cup N_\sigma(d_i)]$$

and define

$$G = \bigcup_{t\leqslant 0} \pi(t, E).$$

On the boundary of G the vectors are nowhere pointing to the interior. By the usual general position arguments, G can be adjusted to give us a neighborhood $G_\sigma(l)$ on whose boundary the vectors point to the exterior.

AN EXAMPLE. As an application of the above let us consider two examples, each on a sphere with h handles Σ^h as illustrated by the configurations shown in Figs. 4 and 5. D is a detractor and S_i are saddles. We make the assumption that there are no transit orbits generated by D or S_i except those shown. Of course, if $h \neq 2$ there must be additional singularities present since the Euler characteristic (which must agree with the index of the vector field) of Σ^h is $2(1-h)$. If $h = 0$, Theorem 1.1 applies to give a periodic orbit in both the cases of Fig. 4 and Fig. 5. If $h = 1$, there must be two "free" detractors and Theorem 2.1 gives us a recurrent motion. Actually we will see shortly that for $h = 1$ we obtain a periodic orbit in the cases of Fig. 4 and Fig. 5. (See Theorem 3.1.) We now consider $h \geqslant 2$. The region inside the dotted circle is topologically a disc, i.e., the figures are distinguished by the fact that in Fig. 5 the transit orbits DS_2 and DS_3 emanate from the same side of the closed loop $l = DS_1D$

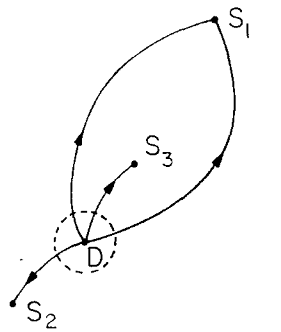

FIGURE 4

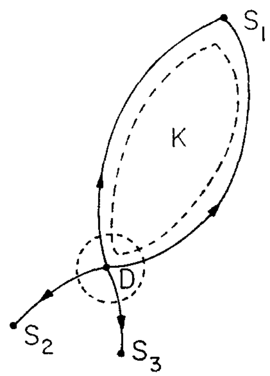

FIGURE 5

By Lemma 3.1 there is a detracting neighborhood of l, $G_\sigma(l)$ that does not contain S_2 nor S_3. If l bounds a disc Δ then the Poincaré–Bendixon theorem implies the existence of a periodic orbit in Δ in both the cases of Fig. 4 and Fig. 5. In the contrary case it is interesting to note that for the flow depicted by Fig. 4 no conclusion can be drawn, while in the case of Fig. 5 there exists a nontrivial recurrent motion on Σ^h. The argument for the latter is as follows. We cut Σ^h along $\partial G_\sigma(l)$, discard the portion containing l and attach to the remaining portion two detracting open discs (a detracting open disc is a small neighborhood of a detractor). The resulting configuration is a (possibly disconnected) compact orientable manifold without boundary. The disc attached to the component K of $\partial G_\sigma(l)$ that fails to intersect each of the transit orbits DS_2 and DS_3 is now a detracting open disc which is free. The remark at the end of Section 2 now is applicable and we have a nontrivial recurrent motion.

The procedure used in the previous example can be applied more generally as follows: Removing a loop and its detracting neighborhood leaves the Euler characteristic of the surface unchanged but possibly disconnects the surface. Attaching two discs raises the characteristic by two. Consider Σ^h with characteristic $\chi = 2(1 - h)$. If there exist h loops then after h steps we have $\chi = 2$ and therefore one of the components must be a sphere. But then Theorem 1.1 applies and we have

THEOREM 3.1. *If there exist h loops on Σ^h and* H1 *and* H2 *hold, then there exists a periodic orbit on Σ^h.*

More generally we may state

THEOREM 3.2. *Let there be loops $l_1, \ldots, l_k$ on Σ^h such that $\Sigma^h - \bigcup_{i=1}^{k} l_k$ has a component Σ^* which is a sphere with holes. Then if* H1 *and* H2 *hold on Σ^*, there exists a periodic orbit on Σ^*.*

The following corollary is now an easy application of the last theorem

COROLLARY. *Consider a flow on the torus Σ^1. Assume* H1 *and* H2 *and that there is at least one detractor. If every detractor generates at least two different transit orbits, then there exists a periodic orbit.*

Remark. This study can be continued by introducing the notion of a *graph* which is defined to be a component of the set

$$E = \left\{\bigcup S_i\right\} \cup \left\{\bigcup T_i\right\},$$

where the union is taken over all singularities S_i and all transit orbits T_i. However this extension is valid not only on 2-manifolds but also for a flow on any compact metric space. These results are discussed in a sequel to the current paper [3].

4. COUNTEREXAMPLES

In this section we shall present two examples which give a negative answer to our basic problem. The first example is a flow on Σ^1 with a single saddle and detractor and one transit orbit. The second example is a flow on Σ^2 with precisely two saddles. For each example, we show that there is no nontrivial recurrent motion even though H1 and H2 hold.

EXAMPLE A (The Pitchfork Flow). We start with a continuous vector field defined on the square in Fig. 6, with a single detractor D and saddle S and a single transit orbit connecting D and S. We shall assume that the vector field is symmetric with respect to the line through D and S, and that the vectors are vertical in the two horizontal strips at the top and bottom of the square. We now identify the vertical sides and get a cylinder Γ, having as boundary the upper circle C_1 and the lower circle C_0. We define a cyclic ordering on C_0 and C_1 so that the positive direction corresponds to the left-to-right ordering in Fig. 6.

Define a mapping $\phi : C_0 \to C_1$ by setting $\phi(P_0) = P_0'$ (see Fig. 6) and for any other $P \in C_0$, $\phi(P)$ is obtained by following the flow. Of course, ϕ is discontinuous at P_0.

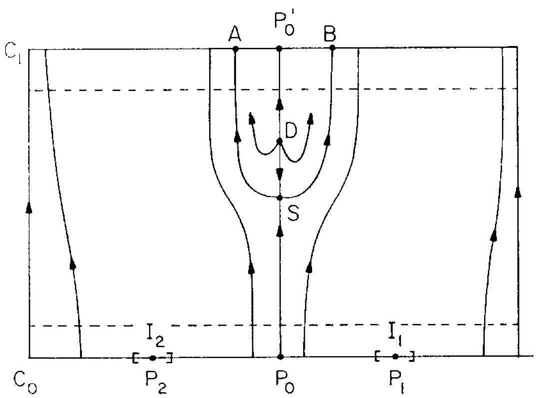

FIGURE 6

Let P_1 and P_2 be two other (distinct) points on C_0 and assume that they are labeled so that $P_0 < P_1 < P_2$ in the ordering on C_0. Let

$$\delta = \min\{d(P_i, P_j) : 0 \leqslant i < j \leqslant 2\}$$

where d denotes the arc-length distance on C_0. Note that $\delta > 0$. Let $\tau_0 = 0$, $\tau_n = \delta \cdot 4^{-n}$, $n = 1, 2, \ldots$, $\sigma_n = 2^{n-3}\tau_n$, $n = 3, 4, \ldots$, and $\sigma_n = \tau_n$ for $n = 0, 1, 2$. One then has

$$\sum_{n=1}^{\infty} \sigma_n < \delta \qquad \text{and} \qquad \sigma_{n+1} < \delta - \{\sigma_1 + \cdots + \sigma_n\}. \tag{4.1}$$

Let $\mathscr{I}$ denote a finite collection of closed intervals in C_0, say $\mathscr{I} = \{I_1, \ldots, I_n\}$. Then $D_0 = C_0 - \{I_1 \cup \cdots \cup I_n\}$ consists of a finite number of open intervals. We shall let $\| \mathscr{I} \|$ denote the arc-length of the smallest open interval in D_0.

We now claim that there is a sequence of points $\{P_0, P_1, \ldots\}$ in C_0 with the following 3 properties:

(1) The closed intervals I_n of arc length τ_n centered at P_n are nonoverlapping.

(2) If $\mathscr{I}_n = \{I_0, \ldots, I_n\}$, then

$$\| \mathscr{I}_n \| \geqslant (\tfrac{1}{2})^{n-2} \{\delta - (\sigma_1 + \cdots + \sigma_n)\}. \tag{4.2}$$

(3) $P_i \leqslant P_j \leqslant P_k$ if and only if $P_{i+1} \leqslant P_{j+1} \leqslant P_{k+1}$.

We prove this by induction. It clearly holds for $\{P_0, P_1, P_2\}$. So now assume that the points $\{P_0, P_1, \ldots, P_k\}$ $(k \geqslant 2)$ have been chosen so that the corresponding intervals $\{I_0, \ldots, I_k\}$ are nonoverlapping, (4.2) holds for $n = k$, and (3) holds for these points.

Let P_l and P_m be the two points from the sequence $\{P_0, P_1, \ldots, P_{k-1}\}$ that are immediately adjacent to P_k in C_0. That is, $P_l < P_k < P_m$ and no other points from $\{P_0, \ldots, P_{k-1}\}$ lie between P_k and the endpoints P_l and P_m. It follows from (3) that the open interval (P_{l+1}, P_{m+1}) contains no points from $\{P_1, \ldots, P_k\}$. (It may contain P_0.) It follows from (2) and (4.1) that the set

$$G_k = (P_{l+1}, P_{m+1}) - \{I_0 \cup I_{l+1} \cup I_{m+1}\}$$

consists of one (or two) open intervals of length (each of length) $> \tau_{n+1}$. (Note that G_k consists of two open intervals if and only if P_0 lies in (P_{l+1}, P_{m+1}).) Let P_{k+1} denote the midpoint of the largest open interval in G_k. (If both intervals have the same length, we choose the open interval adjacent to I_{l+1}, for definiteness.) It follows now that the intervals $\mathscr{I}_{k+1} = \{I_0, \ldots, I_{k+1}\}$ are nonoverlapping. Also

$$\begin{aligned} \| \mathscr{I}_{k+1} \| &\geqslant \tfrac{1}{2} \| \mathscr{I}_k \| - \frac{\tau_{k+1}}{2} \\ &\geqslant (\tfrac{1}{2})^{k-1} \{\delta - (\sigma_1 + \cdots + \sigma_{k+1})\}, \end{aligned}$$

which is (4.2) for $n = k + 1$. Finally, the ordering in (3) is clear by our choice of P_{k+1}. This shows that the sequence of points $\{P_0, P_1, \ldots\}$ and the corresponding intervals $\{I_0, I_1, \ldots\}$ are well-defined and satisfy (1)–(3).

We shall use the fact that $H = \bigcup_{j=1}^{\infty} I_j$ is dense in C_0 in the sequel.

We will now define an orientation preserving mapping h from C_1 to C_0. Let I_0' be the interval $[A, B]$ in C_1 that contains P_0', see Fig. 6. Let

$$h : I_0' \to I_1, \text{ linearly, and define } I_1' = \phi(I_1).$$

In general let

$$h : I_{j-1}' \to I_j, \qquad \text{linearly, and define} \quad I_j' = \phi(I_j), \quad j = 2, 3, \ldots .$$

The mapping h is thus defined on $G = \bigcup_{j=0}^{\infty} I_j' \subset C_1$ and maps this onto $H = \bigcup_{j=1}^{\infty} I_j \subset C_0$. It follows from (3) that h is monotonic on G.

Since H is dense in C_0 and $G = I_0' \cup \phi(H)$, it follows that G is dense in C_1. By its construction, the mapping h is one-to-one on G.

The mapping h can now be extended to C_1, the closure of G, by observing that the right and left hand limits exist for a monotone map. Since the range $H = h(G)$ is dense in C_0, it follows that these limits are the same. Furthermore, this shows that h is continuous, and since C_1 is compact, h is actually a homeomorphism.

We now form a torus T^2 out of the cylinder Γ by identifying a point x on C_1 with $h(x)$ on C_0. This gives rise to a continuous vector field on T^2. We will now show that there are no nontrivial compact minimal sets on T.

Arguing negatively, assume that M is a nontrivial compact minimal set. The set $E = C_0 \cap M$ is then nonempty and compact. Furthermore, since the saddle S is not in M, it follows that P_0 is not in E. Since the detractor D is not in M, it follows that $E \cap I_1 = \varnothing$. Similarly, $E \cap I_n = \varnothing, n = 1, 2, \ldots$. That is $E \cap H = \varnothing$.

Now identify each interval I_j to a single point in I_j, P_j, say. The identification space is again a circle $\tilde{C}_0$. Do the same on C_1 with the intervals I_j' to obtain $\tilde{C}_1$ and note that $\phi : \tilde{C}_0 \to \tilde{C}_1$ is now a homeomorphism onto $\tilde{C}_1$. Note that the identification does not alter the properties of E, i.e., $\tilde{C}_0 - \tilde{E} \neq \varnothing$ where $\tilde{E} = \psi(E)$ and ψ is the identification map. The mapping $T = h \circ \phi : \tilde{C}_0 \to \tilde{C}_0$ is a homeomorphism onto and has a dense orbit $T(P_j) = P_{j+1}$, $j \geqslant 0$. Thus the rotation number of T is irrational [4]. It is then the case that the omega limit set ω_x of the positive semiorbit $\{T^n x\}_{n=0}^{\infty}$ is independent of x. But for $x \in E$, $\omega_x \subset \tilde{E} \neq \tilde{C}_0$ and for $x = P_0$, $\omega_x = \tilde{C}_0$ and therefore a contradiction.

EXAMPLE B. Here we shall construct a flow satisfying hypotheses H1 and H2 on the double torus Σ^2 with precisely two saddle points S_1 and S_2 and no minimal sets other than these saddles. Consider the flow given on the cylinder with two holes shown in Fig. 7, where the vertical boundaries are identified. We will attach a handle to this cylinder by identifying the points on the two interior circles according to the rule $\theta = \phi$, where θ and ϕ are measured in the fashion indicated in Fig. 7. Let ψ_i, $i = 0, 1$, denote the angular measurement on the circles C_i and define a mapping h_ϵ from C_1 to C_0 by setting $\psi_0 = \psi_1 + \epsilon$. This defines a flow on the double torus Σ^2. Define four disjoint open intervals on C_i, $i = 0, 1$, by

$$\begin{aligned} a_i &= \{-\pi/2 < \psi_i < -\pi/4\}, \\ b_i &= \{\pi/4 < \psi_i < \pi/2\}, \\ c_i &= \{-\pi/4 < \psi_i < \pi/4\}, \\ d_i &= \{-\pi \leqslant \psi_i < -\pi/2\} \cup \{\pi/2 < \psi_i \leqslant \pi\}. \end{aligned}$$

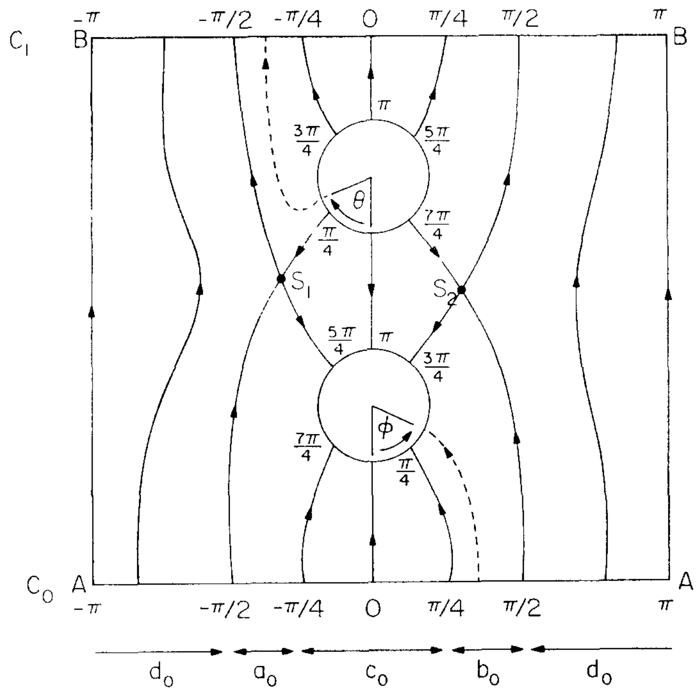

FIGURE 7

We shall assume that the flow on the cylinder with handle induces a mapping ϕ from C_0 to C_1, where

$$\phi : \begin{array}{l} a_0 \to b_1 \\ b_0 \to a_1 \\ c_0 \to c_1 \\ d_0 \to d_1 . \end{array}$$

Furthermore, we shall assume that the flow is adjusted so that ϕ acts linearly on each of these four intervals.

Now let $T_\epsilon = h_\epsilon \circ \phi$. For $\epsilon = 0$, $T_0{}^2$ becomes the identity mapping on each of the intervals a_0, b_0, c_0, and d_0. Now choose ϵ to be in incommensurable with π and consider T_ϵ restricted to the set

$$G = C_0 - \bigcup_{j=1}^{4} \bigcup_{n=0}^{\infty} T_\epsilon^{-n} p_j ,$$

where p_j, $j = 1, 2, 3, 4$, denote the points on C_0 with $\psi_0 = \pm\pi/4, \pm\pi/2$. It is clear from the construction that for each $x \in G$ the set $\{T_\epsilon{}^n x : n = 0, \pm 1, \ldots\}$ is dense in C_0.

If there were a minimal set E in the flow on Σ^2, other than the saddle points, then E must meet C_0 and $E \cap C_0 \subset G$. However, $E \cap C_0$ would then be a closed invariant set under T_ϵ and this is a contradiction. Therefore the only two minimal sets in this flow are the two saddle points

This construction can be extended to flows on Σ^h where $h > 2$ by piecing together $(h - 1)$ copies of this example. That is, in order to build a flow on Σ^3 with these properties take another copy of the cylinder in Fig. 7 and label all its quantities with primes. Define a mapping $g_\nu : C_1 \to C_0'$ by setting $\psi_0' = \psi_1 + \nu$, and a mapping $h_\epsilon : C_1' \to C_0$ by setting $\psi_0 = \psi_1' + \epsilon$, where ν, ϵ and π are incommensurable. Now repeat the previous argument for the mapping

$$T_{\epsilon\nu} = h_\epsilon \circ \phi' \circ g_\nu \circ \phi.$$

This process can obviously be continued.

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